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THE FORMAL SYSTEM FOR VARIOUS 3-VALUED LOGICS I

by

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§1. Introduction and informal discussions.

When we discuss 3-valued logics, we find that some different semantics are introduced. In this paper we shall discuss three logics among them which have much interest related to the theory of computation.

The first one is Kleene's logic introduced in [1]. It is determined by the following four truth tables for propositional connectives.

$A \vee B$

A\B	t	ω	f
t	t	t	t
ω	t	ω	ω
f	t	ω	f

$A \wedge B$

A\B	t	ω	f
t	t	ω	f
ω	ω	ω	f
f	f	f	f

$A \supset B$

A\B	t	ω	f
t	t	ω	f
ω	t	ω	ω
f	t	t	t

$\neg A$

A	−
t	f
ω	ω
f	t

Here t, f or ω means 'true', 'false' or 'undefined' respectively.

Łukasievicz's logic is also described in [1], which differs from Kleene's by the definition of the conditional. It is given by

$A \supset_L B$

A\B	t	ω	f
t	t	ω	f
ω	t	t	ω
f	t	t	t

That is, $A \supset_L A$ is always true in the logic, while it may be undefined in that.

The last one is McCarthy's logic introduced in [2]. It has the same table for the negation as the former, and has the following tables for the other connectives. It is convenient to use the symbols + (or) and . (and) instead of $\vee_M$ and $\wedge_M$.

A+B				A.B				$A \underset{M}{\supset} B$			
A\B	t	ω	f	A\B	t	ω	f	A\B	t	ω	f
t	t	t	t	t	t	ω	f	t	t	ω	f
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
f	t	ω	f	f	f	f	f	f	t	t	t

Here deciding the value of e.g. A+B, we calculate B if necessary, after terminating the calculation of A. Thus if A is undefined, so is A+B regardless of the value of B. While in Kleene's we calculate in pararell, and so when B is true, so is $A \vee B$ even if A is undefined.

The following equalities are easily verified, where = means that the left and the right hand sides always have the same truth value.

$\neg\neg A = A$.

$A \vee A = A$, $A \wedge A = A$

(the absorptivity in Kleene's).

$A+A = A$, $A.A = A$

(the absorptivity in McCarthy's).

$A \vee (B \vee C) = (A \vee B) \vee C$, $A \wedge (B \wedge C) = (A \wedge B) \wedge C$

(the associativity in Kleene's).

$A+(B+C) = (A+B)+C$, $A.(B.C) = (A.B).C$

(the associativity in McCarthy's).

$\neg(A \vee B) = \neg A \wedge \neg B$, $\neg(A \wedge B) = \neg A \vee \neg B$

(de Morgan's property in Kleene's).

$\neg(A+B) = \neg A.\neg B$, $\neg(A.B) = \neg A+\neg B$

(de Morgan's property in McCarthy's)

$A \supset B = \neg A \vee B$,

$A \underset{M}{\supset} B = \neg A+B$,

while $A \underset{L}{\supset} B \neq \neg A \vee B$.

The commutativity in Kleene's holds, i.e.

$$A \vee B = B \vee A, \qquad A \wedge B = B \wedge A,$$

while that in McCarthy's does not hold, i.e.

$$A+B \neq B+A, \qquad A.B \neq B.A.$$

The distributivity in Kleene's and the left distributivity in McCarthy's hold, i.e.

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \qquad A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C),$$

$$(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \qquad (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C),$$

and

$$A+(B.C) = (A+B).(A+C), \qquad A.(B+C) = (A.B)+(A.C),$$

while the right-distributivity in McCarthy's does not hold, i.e.

$$(A.B)+C \neq (A+C).(B+C), \qquad (A+B).C \neq (A.C)+(B.C).$$

McCarthy's connectives can be interpreted in Kleene's by the following equalities.

$$\begin{aligned} A+B &= (A \vee \neg A) \wedge (A \vee B) \\ &= A \vee (\neg A \wedge B), \\ A.B &= (A \wedge \neg A) \vee (A \wedge B) \\ &= A \wedge (\neg A \vee B). \end{aligned}$$

Formulas are constructed in the usual manner. A prime formula is a formula which contains no logical symbol, and a literal is a prime formula or its negation. We call a pair of prime formula and its negation a pair of duals. We use A,B,C, etc. with or without subscript to designate formulas. A sequent is an ordered pair of finite sets of formulas. We use the notation $A_1,\ldots,A_m \Rightarrow B_1,\ldots,B_n$ for a sequent $\langle \{A_1,\ldots,A_m\},\{B_1,\ldots,B_n\} \rangle$. The order of formulas in each side is immaterial. The part $A_1,\ldots,A_m$ is called the antecedent and means the conjunction $A_1 \wedge \ldots \wedge A_m$, and the part $B_1,\ldots,B_n$ is called the succedent and means the disjunction $B_1 \vee \ldots \vee B_n$, where we ragard the empty conjunction as true and the empty disjunction as false. We use Greek capital letters Γ, Δ, Θ, etc. to designate sets of formulas. An assignment t, f or ω to all prime formulas is extended to all formulas in the ordinal way according to the truth tables. Such assignment is said to satisfy a sequent if one of the following conditions is fulfilled: i) it assigns f to some formula in the antecedent, ii) it assigns t to some formula in the succedent, or iii) there are some formulas in both sides to which it assigns ω. In other words the value assigned to the antecedent is less than or equals to the one to the succedent with respect to the order $f<\omega<t$. A sequent is said to be valid if it is satisfied by all assignments.

§2. The formal system for Kleene's 3-valued logic and its plausibility.

In this section we shall describe a formal system, and show its plausibility, that every provable sequent is valid in Kleene's sense.

As axioms we admit the sequents of the form

$$\Gamma, A \Rightarrow \Delta, A \quad ,$$

as well as those of the form

$$\Gamma, A, \neg A \Rightarrow \Delta, B, \neg B \quad .$$

Among the rules of inferences in the propositional calculus of Gentzen's LK, those for conjunction and for disjunction are admitted in our system, while those for negation are refused.

$$(\wedge\Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A\wedge B \Rightarrow \Delta} \quad ,$$

$$(\Rightarrow\wedge) \quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\wedge B} \quad ,$$

$$(\vee\Rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A\vee B \Rightarrow \Delta} \quad ,$$

ans

$$(\Rightarrow\vee) \quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A\vee B} \quad .$$

And the rules for conditional are replaced by the rules

$$(\supset\Rightarrow) \quad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A\supset B \Rightarrow \Delta}$$

and

$$(\Rightarrow\supset) \quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, A\supset B} \quad .$$

Furthermore instead of the rules for negation, we have the rules for double negations and those for combinations of negations and each other symbols which correspond to de Morgan's property.

$$(\neg\neg\Rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \quad ,$$

$$(\Rightarrow\neg\neg) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \quad ,$$

$$(\neg\wedge\Rightarrow) \quad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A\wedge B) \Rightarrow \Delta} \quad ,$$

$$(\Rightarrow\neg\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A\wedge B)}\ ,$$

$$(\neg\vee\Rightarrow)\quad \frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A\vee B) \Rightarrow \Delta}\ ,$$

$$(\Rightarrow\neg\vee)\quad \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A\vee B)}\ ,$$

$$(\neg\supset\Rightarrow)\quad \frac{\Gamma, A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A\supset B) \Rightarrow \Delta}\ ,$$

and

$$(\Rightarrow\neg\supset)\quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A\supset B)}\ .$$

The formula in the lower sequent of a rule, in which the logical symbol is introduced, is called the principal formula of the rule.

The following rules are useful although it shall be clarified later that they are unessential.

$$(\text{cut})\quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Theta, A \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

$$(\text{weakning})\quad \frac{\Gamma \Rightarrow \Delta}{\Theta \Rightarrow \Lambda}$$

where $\Gamma \subset \Theta$ and $\Delta \subset \Lambda$.

A sequent is provable if it is an axiom or the result of applying a rule of inference to sequents which are already known to be provable. A sequent is strictly provable if it is provable without using (cut) nor (weakning).

In order to observe that these rules keep the validity, we must examine it for each rule. Here we shall show it only for the rule $(\Rightarrow\neg\supset)$, for other cases it can be easily shown in the similar way.

Suppose that the lower sequent of the rule $(\Rightarrow\neg\supset)$ were not satisfied by an assignment, and it will be shown that one of the upper sequents is to be not satisfied by the assignment. Since $\neg(A\supset B)$ is in the succedent it takes the value f or ω. When it takes f, either A takes f or B takes t. So the first or the second upper sequent is not satisfied respectively. When $\neg(A \supset B)$ takes ω, Γ must take t, to be exact every formula in Γ takes t. So the first or the second upper sequent is not satisfied according to either A or B takes ω.

Thus the following theorem holds.

PLAUSIBILITY THEOREM. Every provable sequent is valid.

We shall show the formal proof of the equivalence of $(A \supset B) \wedge (B \supset A)$ and $(A \wedge B) \vee (\neg A \wedge \neg B)$ as an example.

$$
\begin{prooftree}
\AxiomC{$\neg A, \neg B \Rightarrow A \wedge B, \neg A$}
\AxiomC{$B, \neg B \Rightarrow A, \neg A$}
\AxiomC{$B, \neg B \Rightarrow B, \neg A$}
\BinaryInfC{$B, \neg B \Rightarrow A \wedge B, \neg A$}
\BinaryInfC{$A \supset B, \neg B \Rightarrow A \wedge B, \neg A$}
\AxiomC{$A \supset B, \neg B \Rightarrow A \wedge B, \neg B$}
\BinaryInfC{$A \supset B, \neg B \Rightarrow A \wedge B, \neg A \wedge \neg B$}
\AxiomC{$A \supset B, A \Rightarrow A, \neg A \wedge \neg B$}
\AxiomC{$\neg A, A \Rightarrow B, \neg A$}
\AxiomC{$\neg A, A \Rightarrow B, \neg B$}
\BinaryInfC{$\neg A, A \Rightarrow B, \neg A \wedge \neg B$}
\AxiomC{$B, A \Rightarrow B, \neg A \wedge \neg B$}
\BinaryInfC{$A \supset B, A \Rightarrow B, \neg A \wedge \neg B$}
\BinaryInfC{$A \supset B, A \Rightarrow A \wedge B, \neg A \wedge \neg B$}
\BinaryInfC{$A \supset B\ ,\ B \supset A \Rightarrow A \wedge B\ ,\ \neg A \wedge \neg B$}
\UnaryInfC{$A \supset B\ ,\ B \supset A \Rightarrow (A \wedge B) \vee (\neg A \wedge \neg B)$}
\UnaryInfC{$(A \supset B) \wedge (B \supset A) \Rightarrow (A \wedge B) \vee (\neg A \wedge \neg B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$A, B \Rightarrow \neg A, B$}
\UnaryInfC{$A, B \Rightarrow A \supset B$}
\UnaryInfC{$A \wedge B \Rightarrow A \supset B$}
\AxiomC{$A, B \Rightarrow \neg B, A$}
\UnaryInfC{$A, B \Rightarrow B \supset A$}
\UnaryInfC{$A \wedge B \Rightarrow B \supset A$}
\BinaryInfC{$A \wedge B \Rightarrow (A \supset B) \wedge (B \supset A)$}
\AxiomC{$\neg A, \neg B \Rightarrow \neg A, B$}
\UnaryInfC{$\neg A, \neg B \Rightarrow A \supset B$}
\UnaryInfC{$\neg A \wedge \neg B \Rightarrow A \supset B$}
\AxiomC{$\neg A, \neg B \Rightarrow \neg B, A$}
\UnaryInfC{$\neg A, \neg B \Rightarrow B \supset A$}
\UnaryInfC{$\neg A \wedge \neg B \Rightarrow B \supset A$}
\BinaryInfC{$\neg A \wedge \neg B \Rightarrow (A \supset B) \wedge (B \supset A)$}
\BinaryInfC{$(A \wedge B) \vee (\neg A \wedge \neg B) \Rightarrow (A \supset B) \wedge (B \supset A)$}
\end{prooftree}
$$

§3. The completeness of the previous system.

We shall show in this section that the system defined in the previous section is complete.

Given a sequent we make the so-called decomposition of it. That is, we construct strings of sequents such that i) the first sequent of the string is the given sequent, ii) when either the n-th sequent is an axiom or it contains only literals, it is the end of the string, and iii) when the n-th sequent is not an axiom and it contains a formula other than literals, the (n+1)-st sequent is one of the upper sequents of the rule whose lower sequent is the n-th sequent and whose principal formula is such a formula mentioned above. It is clear that every string is finite and that if every string ends in an axiom, the given sequent is strictly provable .

Thus making the decomposition of the given sequent, if it is not strictly provable, there is a string containing no axiom. Let Γ or Δ be the set of all formulas which appear in the antecedents or in the succedents of the sequents in the string respectively. Since a literal appearing in a sequent of the string also appears in the same side of every following sequents, i) Γ and Δ have no literal in common, and ii) it is impossible that both Γ and Δ have a pair of duals. We may assume that Δ has no pair of duals. Now we take an assignment, i) which assigns t to prime formulas whose negations are in Δ , ii) which assigns f to those which are in Δ ,and iii) which assigns ω to all other prime formulas.

It is shown by the induction on the number of symbols that the extension of this assignment assigns t or ω to formulas in Γ

and f to those in Δ . We shall show the cases of $A \supset B$. If $A \supset B$ is in Δ ,both $\neg A$ and B are in Δ too. Therefore by the induction hypothesis A takes t and B takes f, so $A \supset B$ takes f. If $A \supset B$ is in Γ, either $\neg A$ or B is in Γ . The former implies that A takes f or ω and $A \supset B$ takes t or ω . The latter implies that B takes t or ω and so is $A \supset B$.

Hence the given sequent is not valid.

The following example will clarify this method.

Suppose that the sequent $(A \supset B) \wedge (\neg A \supset C) \Rightarrow (A \wedge B) \vee (\neg A \wedge C)$ is given. We can find the string:

$$
\begin{array}{ccccc}
(A \supset B) \wedge (\neg A \supset C) & \Rightarrow & (A \wedge B) \vee (\neg A \wedge C) & , \\
(A \supset B) \wedge (\neg A \supset C) & \Rightarrow & A \wedge B \, , \, \neg A \wedge C & , \\
A \supset B \, , \, \neg A \supset C & \Rightarrow & A \wedge B \, , \, \neg A \wedge C & , \\
\neg A \, , \, \neg A \supset C & = & A \wedge B \, , \, \neg A \wedge C & , \\
\neg A \, , \, A & = & A \wedge B \, , \, \neg A \wedge C & , \\
\neg A \, , \, A & = & B \, , \, \neg A \wedge C & , \\
\neg A \, , \, A & = & B \, , \, C & .
\end{array}
$$

So Γ consists of A and $\neg A$ and Δ consists of B and C. Then an assignment which assigns ω to A and f to B and C fails to satisfy the given sequent.

This showed the following theorem.

COMPLETENESS THEOREM. Every valid sequent is strictly provable.

COROLLARY. Every provable sequent is strictly provable.

§4. The formal system for the extended Kleene's 3-valued logic.

We shall give the interpretation of the quantifiers and infinitary connectives in the 3-valued logic.

$(\exists x)A(x)$ is true iff $A(t)$ is true for some term t, it is false iff $A(t)$ is false for every term t, and it is undefined iff none of $A(t)$'s are true and some $A(t)$ is undefined. $\bigwedge(A_1, A_2, \ldots)$ is true iff every A_n is true, it is false iff some A_n is false, and it is undefined iff none of A_n's are false and some A_n is undefined. $(\forall x)A(x)$ and $\bigvee(A_1, A_2, \ldots)$ are interpreted likewise.

The following equalities hold.

$$\neg(\exists x)A(x) = (\forall x)\neg A(x) \quad , \quad \neg(\forall x)A(x) = (\exists x)\neg A(x) \quad ,$$

$$\neg\bigvee(A_1, A_2, \ldots) = \bigwedge(\neg A_1, \neg A_2, \ldots), \neg\bigwedge(A_1, A_2, \ldots) = \bigvee(\neg A_1, \neg A_2, \ldots) \quad ,$$

$$(\exists x)A(x) \vee B = (\exists x)(A(x) \vee B) \quad , \quad (\forall x)A(x) \wedge B = (\forall x)(A(x) \wedge B) \quad ,$$

$$(\exists x)A(x) \wedge B = (\exists x)(A(x) \wedge B) \quad , \quad (\forall x)A(x) \vee B = (\forall x)(A(x) \vee B) \quad ,$$

$$\bigvee(A_1, A_2, \ldots) \vee B = \bigvee(B, A_1, A_2, \ldots) \quad ,$$

$$\bigwedge(A_1, A_2, \ldots) \wedge B = \bigwedge(B, A_1, A_2, \ldots) \quad ,$$

$$\bigvee(A_1, A_2, \ldots) \wedge B = \bigvee(A_1 \wedge B, A_2 \wedge B, \ldots) \quad ,$$

$$\bigwedge(A_1, A_2, \ldots) \vee B = \bigwedge(A_1 \vee B, A_2 \vee B, \ldots) \quad .$$

Rules for those symbols are those in LK with slight modification.

$$(\exists \Rightarrow) \quad \frac{\Gamma, A(a) \Rightarrow \Delta}{, (\exists x)A(x) \Rightarrow \Delta}$$

where a is a free variable not appearing in the lower sequent. Such a is called an eigen-variable.

$$(\Rightarrow \exists) \quad \frac{\Gamma \Rightarrow \Delta, A(t_1), A(t_2), \ldots, A(t_n), (\exists x)A(x) \quad \text{for some } n}{\Gamma \Rightarrow \Delta, (\exists x)A(x)} \quad ,$$

$$(\bigwedge \Rightarrow) \quad \frac{\Gamma, A_1, A_2, \ldots, A_n, \bigwedge(A_1, A_2, \ldots) \Rightarrow \Delta \quad \text{for some } n}{\Gamma, \bigwedge(A_1, A_2, \ldots) \Rightarrow \Delta} \quad ,$$

and

$$(\Rightarrow \bigwedge) \quad \frac{\Gamma \Rightarrow \Delta, A_n \quad \text{for all } n}{\Gamma \Rightarrow \Delta, \bigwedge(A_1, A_2, \ldots)} \quad .$$

In addition we have also rules for combinations of negayions and each of these symbols.

$$(\neg\exists\Rightarrow)\quad \frac{\Gamma,\neg A(t_1),\neg A(t_2),\ldots,\neg A(t_n),\neg(\exists x)A(x)\Rightarrow\Delta \quad \text{for some } n}{\Gamma,\neg(\exists x)A(x)\Rightarrow\Delta}$$

$$(\Rightarrow\neg\exists)\quad \frac{\Gamma\Rightarrow\Delta,\neg A(a)}{\Gamma\Rightarrow\Delta,\neg(\exists x)A(x)}$$

where a is an eigen-variable.

$$(\neg\wedge\Rightarrow)\quad \frac{\Gamma,\neg A_n\Rightarrow\Delta \quad \text{for all } n}{\Gamma,\neg\wedge(A_1,A_2,\ldots)\Rightarrow\Delta},$$

$$(\Rightarrow\neg\wedge)\quad \frac{\Gamma\Rightarrow\Delta,\neg A_1,\neg A_2,\ldots,\neg A_n,\neg\wedge(A_1,A_2,\ldots) \quad \text{for some } n}{\Gamma\Rightarrow\Delta,\neg\wedge(A_1,A_2,\ldots)}.$$

The other rules $(\forall\Rightarrow)$, $(\Rightarrow\forall)$, $(\vee\Rightarrow)$, $(\Rightarrow\vee)$, $(\neg\forall\Rightarrow)$, $(\Rightarrow\neg\forall)$, $(\neg\vee\Rightarrow)$, and $(\Rightarrow\neg\vee)$ are given likewise.

The plausibility of the extended system will be shown by the induction on the number of rules of inferences applied in the proof of the given sequent which is provable and should be shown to be valid.

Suppose that the lower sequent of the rule $(\Rightarrow\exists)$ were not satisfied by an assignment. Then $(\exists x)A(x)$ takes the value f or ω and $A(t_1), A(t_2), \ldots,$ and $A(t_n)$ take also f or ω. Moreover some of them take ω only if $(\exists x)A(x)$ takes ω and then Γ takes t by the assumption. The upper sequent fails to be satisfied by the assignment.

Suppose that the lower sequent of the rule $(\exists\Rightarrow)$ were not satisfied, $(\exists x)A(x)$ takes t or ω. When it takes t, A(t) takes t for some term t. Then the sequent which is obtained from the upper sequent by replacement the eigen-variable a with the term t is not satisfied. This contradicts the induction hypothesis. When $(\exists x)A(x)$ takes ω it is quite similar.

It is shown for the other cases in the similar way.

The completeness of the extended system will be shown using the similar method of decomposition as that of the original system. However there is a little difficulty , and we must devise a means.

Consider an enumeration $t_1, t_2, \ldots$ of all terms and an enumeration of all formulas other than literals.

We construct strings as follows: i) The first sequent of the string is the given sequent. ii) When the n-th sequent is an axiom, it is the end of the string. iii) When the n-th sequent is not an axiom, let i, j be such integers that $n=2^{i-1}(2\cdot j-1)$. iii-i) If it does not contain the i-th formula, the (n+1)-st sequent is the n-th sequent again. iii-ii) If it contains the i-th formula, the (n+1)-st sequent is one of the upper sequents of the rule whose lower sequent is the n-th sequent and whose principal formula is the i-th formula. Moreover when the rule is $(\Rightarrow\exists)$ with the lower sequent $\Gamma \Rightarrow \Delta, (\exists x)A(x)$, the (n+1)-st sequent must be chosen $\Gamma \Rightarrow \Delta, A(t_1), \ldots, A(t_j), (\exists x)A(x)$, that is, all first j terms are substituted to x . When the rule is $(\wedge\Rightarrow)$ with the lower sequent $\Gamma, \wedge(A_1, A_2, \ldots) \Rightarrow \Delta$, the (n+1)-st sequent must be $\Gamma, A_1, \ldots, A_j, \wedge(A_1, A_2, \ldots) \Rightarrow \Delta$. The similar restrictions are also necessary when the rule is $(\neg\exists\Rightarrow)$, $(\forall\Rightarrow)$, $(\Rightarrow\neg\forall)$, $(\Rightarrow\neg\wedge)$, $(\Rightarrow\vee)$, or $(\neg\vee\Rightarrow)$.

If the given sequent is not strictly provable , there is a string which does not end ~~to~~ in an axiom and so is infinite. Let Γ or Δ be sets of formulas in the antecedents or the succedents of the sequents in the string . Also Γ and Δ have no literal in common, and eithr Γ or Δ has no pair of duals. We assume again that Δ has no pair of duals.

Take an assignment i) which assigns t to prime formulas whose negations are in Δ, ii) which assigns f to prime formulas in Δ, and iii) which assigns ω to all other prime formulas. It will be shown by the induction on the number of symbols that every formula in Γ takes t or ω and every formula in Δ takes f. For example, if $(\exists x)A(x)$ is in Δ, then $A(t)$ is in Δ for every term t by means of the construction of the string. So every $A(t)$ takes f and $(\exists x)A(x)$ takes f too.

Therefore the assignment fails to satisfy the given sequent.

§5. The formal system for the extended Łukasiewicz's 3-valued logic.

In this section we shall extend Kleene's logic by adding new symbol $\rightarrow$, called strong conditional, which just reflects the concept of sequents. Łukasiewicz's logic is then interpretable in the extended one.

The truth table for strong conditional is as follows:

$A \rightarrow B$

A\B	t	ω	f
t	t	f	f
ω	t	t	f
f	t	t	t

It is easily seen that $A \rightarrow B$ has always the definit value but ω, and that $\neg B \rightarrow \neg A$ has the same value as it. It is also seen that

$A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is valid if and only if $\Rightarrow A_1 \wedge \ldots \wedge A_m \rightarrow B_1 \vee \ldots \vee B_n$ is valid.

Łukasiewicz's conditional is then interpreted by the following equality:

$$A \underset{L}{\supset} B = (A \supset B) \vee (A \rightarrow B) .$$

Rules for strong conditional are

$$(\rightarrow \Rightarrow)\ \frac{\Gamma, \neg A \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, A, \neg B \quad \Gamma, \neg A, B \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta, \neg B}{\Gamma, A \rightarrow B \Rightarrow \Delta},$$

$$(\Rightarrow \rightarrow)\ \frac{\Gamma, A \Rightarrow \Delta, B \quad \Gamma, \neg B \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, A \rightarrow B},$$

$$(\neg\rightarrow \Rightarrow)\ \frac{\Gamma, A \Rightarrow \Delta, B \quad \Gamma, \neg B \Rightarrow \Delta, \neg A}{\Gamma, \neg(A \rightarrow B) \Rightarrow \Delta},$$

and

$$(\Rightarrow \neg\rightarrow)\ \frac{\Gamma, \neg A \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, A, \neg B \quad \Gamma, \neg A, B \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \rightarrow B)}.$$

The rule

$$\frac{A \Rightarrow B}{\Rightarrow A \rightarrow B}$$

is not necessary since it is covered by the special case of the rule $(\Rightarrow \rightarrow)$ that Γ and Δ are empty, and by the fact:

LEMMA. $\neg A \Rightarrow \neg B$ is provable if and only if $B \Rightarrow A$ is provable. The proof is so easy as omitted.

The plausibilities of these rules are shown by using the table as under, where the rule $(\rightarrow\Rightarrow)$ is taken as an example. The rows of the table are divided corresponding to assignments which do not satisfy the lower sequent of the rule. The last column indicates the upper sequents which fail to be satisfied by the respective assignments. For example the first row asserts that if Γ takes t, Δ takes ω, A takes f, and B takes some value by an assignment, it does not satisfy the first upper sequent.

Γ	Δ	$A \rightarrow B$	A	B	upper sequent
t	ω	t	f	-	first
			-	t	second
			ω	ω	fourth
t	f	t	f	-	first
			-	t	second
			ω	ω	fourth
ω	f	t	f	-	first
			-	t	second
			ω	ω	third

We can prove the completeness of this system in the similar way as in Kleene's.

We can regard $A \supset_L B$ as an abbreviation of $(A \supset B) \vee (A \rightarrow B)$ and we can construct rules for it:

$$(\supset_L\Rightarrow)\ \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A, \neg B}{\Gamma, A \supset_L B \Rightarrow \Delta},$$

$$(\Rightarrow\supset_L)\ \frac{\Gamma, A \Rightarrow \Delta, \neg A, B \qquad \Gamma, \neg B \Rightarrow \Delta, \neg A, B}{\Gamma \Rightarrow \Delta, A \supset_L B},$$

$$(\neg\supset_L\Rightarrow)\ \frac{\Gamma, A, \neg B \Rightarrow \Delta, \neg A \qquad \Gamma, A, \neg B \Rightarrow \Delta, B}{\Gamma, \neg(A \supset_L B) \Rightarrow \Delta},$$

and

$$(\Rightarrow\neg\supset_L)\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, \neg B \qquad \Gamma, \neg A, B \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(A \supset_L B)}.$$

§6. The interpretation of the extended McCarthy's 3-valued logic in Kleene's.

In McCarthy's logic the concept of quantifiers cannot be introduced, while infinitary connectives are allowed as extensions of binary connectives, that is, the value is determined by the serial calculation from the left to the right. We use the notations $\Sigma(A_1,A_2,\ldots)$ and $\Pi(A_1,A_2,\ldots)$ for the infinite disjunction $A_1+A_2+\ldots$ and the infinite conjunction $A_1.A_2\ldots$ respectively.

Formulas in this extended McCarthy's logic are interpreted in Kleene's as follows:

$$\begin{aligned} A+B &= (A\vee\neg A)\wedge(A\vee B) \\ &= A\vee(\neg A\wedge B) \quad, \\ A.B &= (A\wedge\neg A)\vee(A\wedge B) \\ &= A\wedge(\neg A\vee B) \quad, \\ \Sigma(A_1,A_2,\ldots) &= \wedge(\vee(A_1,A_2,\ldots),A_1\vee\neg A_1,A_1\vee A_2\vee\neg A_2,\ldots) \\ &= \vee(A_1,\neg A_1\wedge A_2,\neg A_1\wedge\neg A_2\wedge A_3,\ldots) \quad, \end{aligned}$$

and

$$\begin{aligned} \Pi(A_1,A_2,\ldots) &= \vee(\wedge(A_1,A_2,\ldots),A_1\wedge\neg A_1,A_1\wedge A_2\wedge\neg A_2,\ldots) \\ &= \wedge(A_1,\neg A_1\vee A_2,\neg A_1\vee\neg A_2\vee A_3,\ldots) \quad. \end{aligned}$$

We examine the last equalities.

$\Pi(A_1,A_2,\ldots)$ is t $\Rightarrow$ all A_n's are t

$\Rightarrow$ $\wedge(A_1,A_2,\ldots)$ is t

$\Rightarrow$ $\vee(\wedge(A_1,A_2,\ldots),A_1\wedge\neg A_1,A_1\wedge A_2\wedge\neg A_2,\ldots)$ is t ,

$\Pi(A_1,A_2,\ldots)$ is ω $\Rightarrow$ for some n $A_1,A_2,\ldots$,and A_{n-1} are t and A_n is ω

$\Rightarrow$ $A_1\wedge\ldots\wedge A_n\wedge\neg A_n$ is ω and other components are f or ω

$\Rightarrow$ $\vee(\wedge(A_1,A_2,\ldots),A_1\wedge\neg A_1,A_1\wedge A_2\wedge\neg A_2,\ldots)$ is ω ,

$$\begin{aligned}
\Pi(A_1,A_2,\ldots) \text{ is } f &\Longrightarrow \text{for some } n\ A_1,A_2,\ldots,\text{and } A_{n-1} \text{ are } t \\
&\qquad\quad \text{and } A_n \text{ is } f \\
&\Longrightarrow \text{all components are } f \\
&\Longrightarrow \vee(\wedge(A_1,A_2,\ldots),A_1\wedge\neg A_1,A_1\wedge A_2\wedge\neg A_2,\ldots) \text{ is } f \\
\Pi(A_1,A_2,\ldots) \text{ is } t &\Longrightarrow \text{all } A_n\text{'s are } t \\
&\Longrightarrow \text{all } \neg A_1\vee\ldots\vee\neg A_{n-1}\vee A_n\text{'s are } t \\
&\Longrightarrow \wedge(A_1,\neg A_1\vee A_2,\neg A_1\vee\neg A_2\vee A_3,\ldots) \text{ is } t \quad , \\
\Pi(A_1,A_2,\ldots) \text{ is } \omega &\Longrightarrow \text{for some } n\ A_1,A_2,\ldots,\text{and } A_{n-1} \text{ are } t \\
&\qquad\quad \text{and } A_n \text{ is } \omega \\
&\Longrightarrow \neg A_1\vee\ldots\vee\neg A_{n-1}\vee A_n \text{ is } \omega \text{ and other components} \\
&\qquad\quad \text{are } t \text{ or } \omega \\
&\Longrightarrow \wedge(A_1,\neg A_1\vee A_2,\neg A_1\vee\neg A_2\vee A_3,\ldots) \text{ is } \omega \quad ,
\end{aligned}$$

and

$$\begin{aligned}
\Pi(A_1,A_2,\ldots) \text{ is } f &\Longrightarrow \text{for some } n\ A_1,A_2,\ldots,\text{and } A_{n-1} \text{ are } t \\
&\qquad\quad \text{and } A_n \text{ is } f \\
&\Longrightarrow \neg A_1\vee\ldots\vee\neg A_{n-1}\vee A_n \text{ is } f \\
&\Longrightarrow \wedge(A_1,\neg A_1\vee A_2,\neg A_1\vee\neg A_2\vee A_3,\ldots) \text{ is } f \quad .
\end{aligned}$$

The rule for McCarthy's symbols are obtained in the following manner. At first we replace the principal formula in the lower sequent by the suitable form of the above equalities. Then we apply rules for Kleene's symbols in succession. For example, since

$$\dfrac{\dfrac{\Gamma,A \Rightarrow \Delta \qquad \dfrac{\Gamma,\neg A,B \Rightarrow \Delta}{\Gamma,\neg A\wedge B \Rightarrow \Delta}}{\Gamma,A\vee(\neg A\wedge B) \Rightarrow \Delta}}{\Gamma,\ A+B \Rightarrow \Delta}$$

so

$$(+\Rightarrow)\quad \dfrac{\Gamma,A \Rightarrow \Delta \qquad \Gamma,\neg A,B \Rightarrow \Delta}{\Gamma,\ A+B \Rightarrow \Delta}$$

and since

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta, A, \neg A}{\Gamma \Rightarrow \Delta, A \vee \neg A} \qquad \cfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}}{\cfrac{\Gamma \Rightarrow \Delta, (A \vee \neg A) \wedge (A \vee B)}{\Gamma \Rightarrow \Delta, A+B}}$$

so

$$(\Rightarrow +) \quad \frac{\Gamma \Rightarrow \Delta, A, \neg A \qquad \Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A+B} .$$

And by the de Morgan's property

$$(\neg + \Rightarrow) \quad \frac{\Gamma, A, \neg A \Rightarrow \Delta \qquad \Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A+B) \Rightarrow \Delta}$$

and

$$(\Rightarrow \neg +) \quad \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A+B)} .$$

Rules for infinitary disjunction are

$$(\Sigma \Rightarrow) \quad \frac{\Gamma, \neg A_1, \neg A_2, \ldots, \neg A_{n-1}, A_n \Rightarrow \Delta \quad \text{for all } n}{\Gamma, \Sigma(A_1, A_2, \ldots) \Rightarrow \Delta} ,$$

$$(\Rightarrow \Sigma) \quad \frac{\Gamma \Rightarrow \Delta, A_1, \neg A_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, A_1, \ldots, A_{n-1}, \neg A_{n-1} \qquad \Gamma \Rightarrow \Delta, \Sigma(A_1, A_2, \ldots), A_1, \ldots, A_n \quad \text{for some } n}{\Gamma \Rightarrow \Delta, \Sigma(A_1, A_2, \ldots)} ,$$

$$(\neg \Sigma \Rightarrow) \quad \frac{\Gamma, \neg A_1, A_1 \Rightarrow \Delta \quad \ldots \quad \Gamma, \neg A_1, \ldots, \neg A_{n-1}, A_{n-1} \Rightarrow \Delta \qquad \Gamma, \neg\Sigma(A_1, A_2, \ldots), \neg A_1, \ldots, \neg A_n \Rightarrow \Delta \quad \text{for some } n}{, \quad (A_1, A_2, \ldots) =} ,$$

and

$$(\Rightarrow \neg \Sigma) \quad \frac{\Gamma \Rightarrow \Delta, A_1, A_2, \ldots, A_{n-1}, \neg A_n \quad \text{for all } n}{\Gamma \Rightarrow \Delta, \neg\Sigma(A_1, A_2, \ldots)} .$$

The rule $(\Rightarrow \Sigma)$ has the undefinite number of upper sequents. We can avoid it if we leave $\bigvee(A_1, A_2, \ldots)$ in the succedent of an upper sequent. However we choose this form to exclude the symbol $\bigvee$.

The following examples prove the equivalence of A.(B+C) and (A.B)+(A.C)

```
         A,B ⇒ A   A,B ⇒ ¬A,B     A,¬B ⇒ A,¬A   A,¬B ⇒ ¬A, ¬B
         ---------------------    ----------------------------
              A,B ⇒ A.B                 A,¬B ⇒ ¬(A.B)
            -----------------------------------------------
            A,B ⇒ A.B,¬(A.B)    A,¬B,C ⇒ A.B,¬(A.B)
            -----------------------------------------------
A,¬A = A.B,¬(A.B)              A,B+C ⇒ A.B,¬(A.B)
-----------------------------------------------------------
        A.(B+C) ⇒ A.B,¬(A.B)
           :                                       @                  @
           :       A,¬A ⇒ A    A,¬A ⇒ ¬A,B    A,B ⇒ A.B          A,C ⇒ A.C
           :       ------------------------   -----------    -----------------
           :             A,¬A = A.B           A,B ⇒ A.B,A.C   A,¬B,C ⇒ A.B,A.C
           :             ----------           ---------------------------------
           :             A,¬A ⇒ A.B,A.C              A,B+C ⇒ A.B,A.C
           :             -----------------------------------------------
           :                         A.(B+C) ⇒ A.B,A.C
        ---------------------------------------------------------
                  A.(B+C) ⇒ (A.B)+(A.C)

                   A,¬A ⇒ ¬A,B   A,B ⇒ ¬A,B    B ⇒ B,¬B   B ⇒ B,C
                   -------------------------   ------------------
A,¬A ⇒ A   A,B ⇒ A         A.B ⇒ ¬A,B                B ⇒ B+C
-------------------        ---------------------------------------  (cut)
     A.B ⇒ A                          A.B ⇒ ¬A,B+C
     ---------------------------------------------
          A.B ⇒ A.(B+C)
             :
             :           ¬A ⇒ ¬A,¬B   A,¬B ⇒ ¬A,¬B                    @
             :           --------------------------
             :                 ¬(A.B) ⇒ ¬A,¬B                     A.C ⇒ ¬A,C
             :        @     ---------------------------    ------------------------
             :    A.C ⇒ A   ¬(A.B),A.C ⇒ ¬A,B,¬B           ¬(A.B),A.C ⇒ ¬A,B,C
             :  ----------  -----------------------------------------------------
             :  ¬(A.B),A.C ⇒ A          ¬(A.B),A.C ⇒ ¬A,(B+C)
             :  -------------------------------------------------
             :              ¬(A.B),A.C ⇒ A.(B+C)
        -----------------------------------------------
               (A.B)+(A.C) ⇒ A.(B+C)
```

In the proof we omitt the upper sequents of some sequents, which are marked @ . They are easily fulfilled since the same or similar sequents occur before with the upper sequents.

§7. Correlation between the present formulation and Takahashi's.

As above we formulate the system by using the instrument 'sequent' , by the way Takahashi used 'matrix' in his general method to formulate many valued logics.

Now we shall apply his method to Kleene's logic and compare it with ours.

A matrix is an ordered triple of finite sets of formulas, for which we use the notation $\{A_1,\ldots,A_l\}_f \cup \{B_1,\ldots,B_m\}_\omega \cup \{C_1,\ldots,C_n\}_t$. The part $A_1,\ldots,A_l$ is called the f-part, $B_1,\ldots,B_m$ the ω-part, and $C_1,\ldots,C_n$ the t-part.

An assignment satisfies a matrix iff it assigns f to some formula in the f-part, ω to some in the ω-part, or t to some in the t-part.

Axioms are matrices of the form $\{\Gamma, A\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta, A\}_t$.

Rules for the negation are

$$(\neg f)\quad \frac{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, A\}_t}{\{\Gamma, \neg A\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}\ ,$$

$$(\neg \omega)\quad \frac{\{\Gamma\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Theta, \neg A\}_\omega \cup \{\Delta\}_t}\ ,$$

and

$$(\neg t)\quad \frac{\{\Gamma, A\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, \neg A\}_t}\ .$$

Rules for the conjunction are

$$(\wedge f)\quad \frac{\{\Gamma, A, B\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}{\{\Gamma, A \wedge B\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}\ ,$$

$$(\wedge \omega)\quad \frac{\{\Gamma\}_f \cup \{\Theta, A, B\}_\omega \cup \{\Delta\}_t \quad \{\Gamma\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta, A\}_t \quad \{\Gamma\}_f \cup \{\Theta, B\}_\omega \cup \{\Delta, B\}_t}{\{\Gamma\}_f \cup \{\Theta, A\ B\}_\omega \cup \{\Delta\}_t}\ ,$$

and

$$(\wedge t)\quad \frac{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, A\}_t \quad \{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, B\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, A\ B\}_t}\ .$$

The rule $(\wedge f)$, say, originally consists of five schmata and they are condensed to one schema descrived above.

Rules for the existential quantifier are

$$(\exists f)\quad \frac{\{\Gamma, A(a)\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}{\{\Gamma, (\exists x)A(x)\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}$$

where a is an eigen-variable,

$$(\exists\omega)\quad \frac{\{\Gamma, A(a)\}_f \cup \{\Theta, A(a)\}_\omega \cup \{\Delta\}_t \qquad \{\Gamma\}_f \cup \{\Theta, A(t)\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Theta, (\exists x)A(x)\}_\omega \cup \{\Delta\}_t}$$

where a is an eigen-variable and t is a term,

and $$(\exists t)\quad \frac{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, A(t)\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta, (\exists x)A(x)\}_t}$$

where t is a term.

Rules for other symbols, $(\vee f)$, $(\vee\omega)$, $(\vee t)$, $(\supset f)$, $(\supset\omega)$, $(\supset t)$, $(\forall f)$, $(\forall\omega)$, and $(\forall t)$, are omitted.

The following rules are not essential but useful.

$$(\mathrm{cut})\quad \frac{\{\Gamma, A\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t \qquad \{\Gamma\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta, A\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t},$$

$$(\mathrm{cut})\quad \frac{\{\Gamma\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta\}_t \qquad \{\Gamma, A\}_f \cup \{\Theta\}_\omega \cup \{\Delta, A\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t},$$

$$(\mathrm{cut})\quad \frac{\{\Gamma\}_f \cup \{\Theta\} \cup \{\Delta, A\}_t \qquad \{\Gamma, A\}_f \cup \{\Theta, A\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t},$$

and $$(\mathrm{weakning})\quad \frac{\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t}{\{\Gamma'\}_f \cup \{\Theta'\}_\omega \cup \{\Delta'\}_t}$$

where $\Gamma \subset \Gamma'$, $\Theta \subset \Theta'$, and $\Delta \subset \Delta'$.

The following theorem states the correlation between his system and ours.

THEOREM. A sequent $\Gamma \Rightarrow \Delta$ is provable in ours if and only if the matrices $\{\Gamma\}_f \cup \{\Gamma\}_\omega \cup \{\Delta\}_t$ and $\{\Gamma\}_f \cup \{\Delta\}_\omega \cup \{\Delta\}_t$ are both provable in his. Conversely a matrix $\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t$ is provable in his if and only if the sequents $\Gamma, \overline{\Delta}, \Theta_\psi \Rightarrow \overline{\Theta}_\psi$ are provable in ours, equivalently the sequents $\Theta_\psi \Rightarrow \overline{\Gamma}, \Delta, \overline{\Theta}_\psi$ are provable in ours, for all mapping ψ from

Θ to $\{0,1\}$, where Θ_ψ (or $\overline{\Theta}_\psi$) is the set of formulas A's such that $\psi(A)=0$ (or $\psi(A)=1$) and of $\neg A$'s such that $\psi(A)=1$ (or $\psi(A)=0$), and $\overline{\Gamma}$ or $\overline{\Delta}$ consists of the negations of formulas in Γ or Δ respectively .

Since both systems are complete, it is sufficient to prove the theorem semantically.

It is easily seen that an assignment satisfies $\Gamma \Rightarrow \Delta$ iff it satisfies $\{\Gamma\}_f \cup \{\Gamma\}_\omega \cup \{\Delta\}_t$ and $\{\Gamma\}_f \cup \{\Delta\}_\omega \cup \{\Delta\}_t$. In fact they are both equivalent to the statement : the assignment assigns f to some formula in Γ , t to some in Δ , or ω to some in Γ and to some in Δ .

It is also easily seen that an assignment which satisfies $\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t$ also satisfies $\Gamma, \overline{\Delta}, \Theta_\psi \Rightarrow \overline{\Theta}_\psi$ and $\Theta_\psi \Rightarrow \overline{\Gamma}, \Delta, \overline{\Theta}_\psi$ for all ψ. Conversely suppose that an assignment satisfies $\Gamma, \overline{\Delta}, \Theta_\psi \Rightarrow \overline{\Theta}_\psi$ for all ψ . If it assigns ω to some formula in θ , it clearly satisfies $\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t$ too. Otherwise take the mapping ψ such that $\psi(A)$ is 0 or 1 according as A takes the value t or f, then we find that some formula in $\Gamma \cup \overline{\Delta}$ must take f. Therefore $\{\Gamma\}_f \cup \{\Theta\}_\omega \cup \{\Delta\}_t$ is satisfied.

We note that an axiom $A \Rightarrow A$ in ours corresponds to $\{A\}_f \cup \{A\}_\omega \cup \{A\}_t$ which is also an axiom in his, and that an axiom $A, \neg A \Rightarrow B, \neg B$ corresponds to $\{A,\neg A\}_f \cup \{A,\neg A\}_\omega \cup \{B,\neg B\}_t$ and $\{A,\neg A\}_f \cup \{B,\neg B\}_\omega \cup \{B,\neg B\}_t$ which are provable in his as follows:

$$\frac{\{A\}_f \cup \{A,\neg A\}_\omega \cup \{A,B,\neg B\}_t}{\{A,\neg A\}_f \cup \{A,\neg A\}_\omega \cup \{B,\neg B\}_t}$$

and

$$\frac{\{A,\neg A,B\}_f \cup \{B,\neg B\}_\omega \cup \{B\}_t}{\{A,\neg A\}_f \cup \{B,\neg B\}_\omega \cup \{B,\neg B\}_t}$$

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